

SOME CONJECTURES ON GENERALIZED CLUSTER ALGEBRAS VIA THE CLUSTER FORMULA AND D -MATRIX PATTERN

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ABSTRACT. In the theory of generalized cluster algebras, we build the so-called cluster formula and D -matrix pattern. Then as applications, some fundamental conjectures of generalized cluster algebras are solved affirmatively.

1. INTRODUCTION

Cluster algebras were introduced by Fomin and Zelevinsky in [8]. The motivation was to create a common framework for phenomena occurring in connection with total positivity and canonical bases. By now numerous connections between cluster algebras and other branches of mathematics have been discovered, e.g. the theory of quiver representations, categorifications over some important algebras and Poisson geometry, etc..

There are many interesting conjectures about cluster algebras, for example, as follows. Note that in this paper, *the positive integer n always denotes the rank of a cluster algebra*.

Conjecture 1.1. ([10, 14]) (a) *The exchange graph of a cluster algebra with rank n only depends on the initial exchange matrix;*

(b) *Every seed is uniquely determined by its cluster under mutation equivalence;*

(c) *Two clusters are adjacent in the exchange graph if and only if they have exactly $n - 1$ common cluster variables.*

In [14], M. Gekhtman, M. Shapiro and A. Vainshtein proved the following facts in the skew-symmetrizable case for standard cluster algebras:

(1) (a) is true for B with full rank.

(2) (b) implies (c).

(3) (b) is true for cluster algebras of geometric type, and for cluster algebras whose exchange matrix is of full rank.

It is also known that (b) is true for cluster algebras having some “realization”, for example cluster algebras from surfaces [7] and the cluster algebra which has a categorification [2, 6, 1].

We know a cluster should contain the whole information of the corresponding exchange matrix under the assumption that (b) is true. Trivially, (b) implies the following statement (d):

(d) *The exchange matrix could be uniquely recovered from a given cluster.*

In this paper, our aim is to discuss the above conjectures for generalized cluster algebras.

Generalized cluster algebras were introduced in [5] by Chekhov and Shapiro, which are the generalization of the (standard) cluster algebras introduced by Fomin and Zelevinsky in [8]. In the standard case, a product of cluster variables, one known and one unknown, is equal to a binomial in other known variables. These binomial exchange relations is replaced by polynomial exchange relations in generalized cluster algebras. The structure of generalized cluster algebras naturally appears

from the Teichmüller spaces of Riemann surfaces with orbifold points [5]. It also is raised in representations of quantum affine algebras [12] and in WKB analysis [16]. It can be seen in [5, 17] that many important properties and definitions of the standard cluster algebras are naturally extended to the generalized ones, for examples, Laurent phenomenon, finite type classification, the **c**-vectors, **g**-vectors and **F-polynomials**. From these views, we know generalized cluster algebra is an essential improvement of the standard cluster algebras.

We want to consider Conjecture 1.1 in the case of generalized cluster algebra. As a tool for studying generalized cluster algebras including cluster algebras, we give the so-called **cluster formula**. Relying the using of the cluster formula, our method will be constructive, in particular, to recover the exchange matrix from a given cluster, as a direct proof of (d). Finally, we will show that Conjecture 1.1 holds for generalized cluster algebra in general case.

Cluster algebras is introduced in a general case, which means their coefficients are in a general semifield. But it seems that many researchers are more interested in cluster algebras of geometric type. Many conjectures were proved for such cluster algebras, for example, Conjecture 1.1 given above. However, these conjectures are also believed true for cluster algebras with general coefficients. In order to consider Conjecture 1.1 in the case of cluster algebras with general coefficients, we introduce the **D-matrix pattern**, which explains the connection between any two (generalized) cluster algebras having the same initial exchange matrix with different coefficient rings from the view of exchange graphs. More precisely, we give a positive answer to Conjecture 1.1 for (generalized) cluster algebras with any coefficients, whose restricted results on standard cluster algebras are also an improvement of the conclusions in the case of geometrical type, given by other mathematicians early.

This paper is organized as follows: in Section 2, some basic definitions are needed. In Section 3, we give the cluster formula, which is a main result in this paper. As an application, we prove (b) in Conjecture 1.1 is true for generalized cluster pattern of weak geometric type. In the final part of the Section 3, we give the connection between cluster formula and compatible 2-form. In Section 4, we give a positive answer to Conjecture 1.1 in the case of generalized cluster algebra.

2. PRELIMINARIES

We know that $(\mathbb{P}, \oplus, \cdot)$ is a **semifield** if (\mathbb{P}, \cdot) is an abelian multiplicative group endowed with a binary operation of auxiliary addition \oplus which is commutative, associative, and distributive with respect to the multiplication \cdot in \mathbb{P} .

Let $Trop(u_i : i \in I)$ be a free abelian group generated by $\{u_i : i \in I\}$ for a finite set of index I . We define the addition \oplus in $Trop(u_i : i \in I)$ by $\prod_i u_i^{a_i} \oplus \prod_i u_i^{b_i} = \prod_i u_i^{\min(a_i, b_i)}$, then $(Trop(u_i : i \in I), \oplus)$ is a semifield, which is called a **tropical semifield**.

The multiplicative group of any semifield \mathbb{P} is torsion-free for multiplication [8], hence its group ring $\mathbb{Z}\mathbb{P}$ is a domain.

The following proposition can be checked directly:

Proposition 2.1. *Assume $\mathbb{P}_1, \mathbb{P}_2$ are two semifield, let $\mathbb{P} = \mathbb{P}_1 \amalg \mathbb{P}_2 = \{(p_1, p_2) | p_1 \in \mathbb{P}_1, p_2 \in \mathbb{P}_2\}$. Then \mathbb{P} is a semifield via $(p_1, p_2) \cdot (\bar{p}_1, \bar{p}_2) := (p_1 \cdot \bar{p}_1, p_2 \cdot \bar{p}_2)$ and $(p_1, p_2) \oplus (\bar{p}_1, \bar{p}_2) := (p_1 \oplus \bar{p}_1, p_2 \oplus \bar{p}_2)$.*

Definition 2.2. (i) A square integer matrix $B = (b_{ij})_{n \times n}$ is called **skew-symmetric** if $b_{ij} = -b_{ji}$ for any $i, j = 1, \dots, n$; (ii) In general, $B = (b_{ij})_{n \times n}$ is called **skew-symmetrizable** if there exists a diagonal matrix T with positive integer diagonal entries s_1, \dots, s_n such that TB is skew-symmetric.

We take an ambient field \mathcal{F} to be the field of rational functions in n independent variables with coefficients in $\mathbb{Z}\mathbb{P}$.

Definition 2.3. [8, 10, 17] A **(labeled) seed** Σ in \mathcal{F} is a triplet (X, Y, B) such that

(i) $X = (x_1, \dots, x_n)$ is an n -tuple with n algebraically independent variables x_1, \dots, x_n over $\mathbb{Z}\mathbb{P}$.

We call X a **cluster** and x_1, \dots, x_n **cluster variables**.

(ii) $Y = (y_1, \dots, y_n)$ is an n -tuple of elements in \mathbb{P} , where y_1, \dots, y_n are called **coefficients**.

(iii) $B = (b_{ij})$ is an $n \times n$ integer skew-symmetrizable matrix, called an **exchange matrix**.

Let (R, Z) be a pair with $R = (r_i)_{n \times n}$ a diagonal matrix, $r_i \in \mathbb{N}$, and $Z = (z_{i,m})_{i=1, \dots, n; m=1, \dots, r_i-1}$ a family of elements in \mathbb{P} satisfying the reciprocity condition $z_{i,m} = z_{i, r_i-m}$ for $m = 1, \dots, r_i - 1$. And, denote the notations $z_{i,0} = z_{i, r_i} = 1$ for $i = 1, \dots, n$.

Definition 2.4. ([17]) Let $\Sigma = (X, Y, B)$ be a seed in \mathcal{F} , we define the (R, Z) -**mutation** $\mu_k(\Sigma) = (\bar{X}, \bar{Y}, \bar{B})$ of Σ in the direction $k \in \{1, \dots, n\}$ as a new seed in \mathcal{F} :

$$(1) \quad \bar{x}_i = \begin{cases} x_i, & \text{if } i \neq k \\ x_k^{-1} \left(\prod_{j=1}^n x_j^{[-b_{jk}]_+} \right)^{r_k} \left(\sum_{m=0}^{r_k} z_{k,m} \hat{y}_k^m \right) / \left(\bigoplus_{m=0}^{r_k} z_{k,m} y_k^m \right), & \text{if } i = k. \end{cases}$$

$$(2) \quad \bar{y}_i = \begin{cases} y_k^{-1}, & i = k \\ y_i \left(y_k^{[b_{ki}]_+} \right)^{r_k} \left(\bigoplus_{m=0}^{r_k} z_{k,m} y_k^m \right)^{-b_{ki}}, & \text{otherwise.} \end{cases}$$

$$(3) \quad \bar{b}_{ij} = \begin{cases} -b_{ij}, & i = k \text{ or } j = k \\ b_{ij} + r_k(b_{ik}[-b_{kj}]_+ + [b_{ik}]_+ b_{kj}), & \text{otherwise.} \end{cases}$$

for $i, j = 1, 2, \dots, n$, where $[a]_+ = \max\{a, 0\}$, $\hat{y}_i = y_i \prod_{j=1}^n x_j^{b_{ji}}$.

Remark 2.5. (i). It is easy to check that the (R, Z) -mutation μ_k is an involution.

(ii). If $R = I_n$, then (3) is the standard matrix mutation. Let B' be the matrix obtained from BR by the standard matrix mutation in the direction k , it is easy to see $B' = \bar{B}R$. We can write $\mu_k(BR) = \mu_k^g(B)R$, where $\mu_k^g(B) = \bar{B}$ with μ_k^g called the **generalized matrix mutation**.

Definition 2.6. ([17]) An (R, Z) -**cluster pattern** (or say, **generalized cluster pattern**) M in \mathcal{F} is an assignment for each seed Σ_t to a vertex t of the n -regular tree \mathbb{T}_n , such that for any edge $t \xrightarrow{k} t'$, $\Sigma_{t'} = \mu_k(\Sigma_t)$. The triple of Σ_t are written as follows:

$$X_t = (x_{1;t}, \dots, x_{n;t}), \quad Y_t = (y_{1;t}, \dots, y_{n;t}), \quad B_t = (b_{ij}^t).$$

Remark 2.7. (i) Clearly, for each vertex of \mathbb{T}_n , we can uniquely determine the (R, Z) -cluster pattern under (R, Z) -mutations.

(ii) When $R = I_n$ the identity matrix, Z must be empty. In this case, the generalized cluster pattern is just the standard cluster pattern.

Definition 2.8. Let M be an (R, Z) -cluster pattern, we denote by $\mathcal{X} = \{x_{i;t} : t \in \mathbb{T}_n, 1 \leq i \leq n\}$ the set of all cluster variables. The **generalized cluster algebra** \mathcal{A} associated with a given (R, Z) -cluster pattern is the $\mathbb{Z}\mathbb{P}$ -subalgebra of the field \mathcal{F} generated by all cluster variables, i.e. $\mathcal{A} = \mathbb{Z}\mathbb{P}[\mathcal{X}]$.

By definition, \mathcal{A} can be obtained from any given seed Σ_{t_0} for $t_0 \in \mathbb{T}_n$ via mutations. So, we denote $\mathcal{A} = \mathcal{A}(\Sigma_{t_0})$ and call Σ_{t_0} the **initial seed** of \mathcal{A} .

Definition 2.9. (Restriction) (i) Let J be a subset of $\langle n \rangle = \{1, 2, \dots, n\}$. Remove from \mathbb{T}_n all edges labeled by indices in $\langle n \rangle \setminus J$, and denote $\mathbb{T}_n^{t_0}(J)$ the connected component of the resulting graph containing the vertex t_0 in \mathbb{T}_n . Say that $\mathbb{T}_n^{t_0}(J)$ is obtained from \mathbb{T}_n by restriction to J including t_0 . Trivially, $\mathbb{T}_n^{t_0}(J)$ is a $|J|$ -regular tree.

(ii) Let M be an (R, Z) -cluster pattern on \mathbb{T}_n in \mathcal{F} with the seed $\Sigma_t = (X_t, Y_t, B_t)$. We define a **restricted generalized cluster pattern** $M^{t_0}(J)$ on $\mathbb{T}_n^{t_0}(J)$ by assigning the seed $\Sigma_t^{t_0}(J) = (\bar{X}_t, \bar{Y}_t, \bar{B}_t)$ at $t \in \mathbb{T}_n^{t_0}(J)$ with $\bar{X}_t = (x_{j;t})_{j \in J}$, $\bar{Y}_t = (y_{j;t})_{j \in J}$, $\bar{B}_t = (b_{ij}^t)_{i,j \in J}$. Actually, $M^{t_0}(J)$ is a (\bar{R}, \bar{Z}) -cluster pattern on $\mathbb{T}_n^{t_0}(J)$ in the semifield $\mathbb{P} \amalg \text{Trop}(x_{i;t_0} : i \in \langle n \rangle \setminus J)$, where $\bar{R} = \text{diag}\{r_j\}_{j \in J}$, $\bar{Z} = (z_{j,m})_{j \in J, m=1, \dots, r_j-1}$. Say that $M^{t_0}(J)$ is obtained from M by restriction to J including t_0 .

Note that in the view in [15], we can think the seed $\Sigma_t^{t_0}(J)$ as a mixing-type subseed of Σ_t , that is, $\Sigma_t^{t_0}(J) = (\Sigma_t)_{\langle n \rangle \setminus J, \emptyset}$.

Definition 2.10. (a) Assume \mathbb{P}_1 is a semifield, $\mathbb{P}_2 = \text{Trop}(u_j : j \in I)$, where $h = |I| < +\infty$. An (R, Z) -cluster pattern M with coefficients in $\mathbb{P} = \mathbb{P}_1 \amalg \mathbb{P}_2$ is said to be of **weakly geometric type** if the following hold:

- (i) Z is a family of elements in \mathbb{P}_1 .
- (ii) $y_{i;t}$ is a Laurent monomial and denote it by $y_{i;t} = u_1^{c_{1i}^t} u_2^{c_{2i}^t} \dots u_h^{c_{hi}^t}$.
- (b) Further, if $\mathbb{P}_1 = \text{Trop}(Z)$, where we regard $Z = (z_{i,m})_{i=1, \dots, n; m=1, \dots, r_i-1}$ with $z_{i,m} = z_{i, r_i-s}$ as formal variables, then we say M to be an (R, Z) -cluster pattern of **geometric type**.

Proposition 2.11. Let M be a (R, Z) -cluster pattern of weakly geometric type, $Y_t = (y_{1,t}, \dots, y_{n,t})$ be the coefficient tuples at t , where $y_{i;t} = u_1^{c_{1i}^t} u_2^{c_{2i}^t} \dots u_h^{c_{hi}^t}$. Define $C_t = (c_{ij}^t)$. Then for any edge $t \xrightarrow{k} \bar{t}$ in \mathbb{T}_n , C_t and $C_{\bar{t}}$ are related by the **formula of mutation of C -matrices**:

$$\bar{c}_{ij}^t = \begin{cases} -c_{ij}^t, & \text{if } j = k; \\ c_{ij}^t + r_k(c_{ik}^t[b_{kj}^t]_+ + [-c_{ik}^t]_+ b_{kj}^t), & \text{otherwise.} \end{cases}$$

Proof. By (2), we have

$$u_1^{\bar{c}_{1i}^t} \dots u_h^{\bar{c}_{hi}^t} = \begin{cases} (u_1^{c_{1k}^t} u_2^{c_{2k}^t} \dots u_h^{c_{hk}^t})^{-1}, & \text{if } i = k; \\ u_1^{c_{1i}^t} \dots u_h^{c_{hi}^t} \left((u_1^{c_{1k}^t} \dots u_h^{c_{hk}^t})^{[b_{ki}^t]_+} \right)^{r_k} \left(\bigoplus_{m=0}^{r_k} z_{k,m} (u_1^{c_{1k}^t} \dots u_h^{c_{hk}^t})^m \right)^{-b_{ki}}, & \text{otherwise.} \end{cases}$$

Then, we obtain $\bigoplus_{m=0}^{r_k} z_{k,m} = 1$ for $k = 1, \dots, n$ and

$$\bar{c}_{ij}^t = \begin{cases} -c_{ij}^t, & \text{if } j = k; \\ c_{ij}^t + r_k(c_{ik}^t[b_{kj}^t]_+ + [-c_{ik}^t]_+ b_{kj}^t), & \text{otherwise.} \end{cases}$$

□

Definition 2.12. We say M to be an (R, Z) -cluster pattern with (weakly) **principle coefficients** at t_0 , if M is of (weakly) geometric type on \mathbb{T}_n and $C_{t_0} = I_n$.

Remark 2.13. The definition of (R, Z) -cluster pattern with principle coefficients given here is the same with the one in [17].

3. CLUSTER FORMULA AND RELATED RESULTS

3.1. The cluster formula.

A fundamental fact is that for any vertex $t \in \mathbb{T}_n$ and the corresponding seed $\Sigma_t = (X_t, Y_t, B_t)$, RB_t is always a skew-symmetrizable matrix, that is, there is a positive integer diagonal matrix S such that SRB_t is skew-symmetric.

Indeed, since B_t is skew-symmetrizable, we have a positive integer diagonal matrix T which does not depend on t such that TB_t is a skew-symmetric matrix. For $R = (r_i)$, let $S = (\prod_{i=1}^n r_i)TR^{-1}$. Trivially, S is a positive integer diagonal matrix. Then $SRB_t = (\prod_{i=1}^n r_i)TR^{-1}RB_t = (\prod_{i=1}^n r_i)TB_t$ is skew-symmetric.

The diagonal matrix $S = (\prod_{i=1}^n r_i)TR^{-1}$ will be valuable for the following discussion, which is called the **R -skew-balance** for all seeds Σ_t with $t \in \mathbb{T}_n$.

Let $\Sigma_t = (X_t, Y_t, B_t)$, $\Sigma_{t_0} = (X_{t_0}, Y_{t_0}, B_{t_0})$ be two seeds of M at t and t_0 . Considering Σ_{t_0} as the initial seed, we know $x_{i;t}$ is a rational function in $x_{1;t_0}, \dots, x_{n;t_0}$ with coefficients in $\mathbb{Z}\mathbb{P}$ for each i .

Let

$$J_{t_0}^t(X) = \begin{pmatrix} \frac{\partial x_{1;t}}{\partial x_{1;t_0}} & \frac{\partial x_{2;t}}{\partial x_{1;t_0}} & \cdots & \frac{\partial x_{n;t}}{\partial x_{1;t_0}} \\ \frac{\partial x_{1;t}}{\partial x_{2;t_0}} & \frac{\partial x_{2;t}}{\partial x_{2;t_0}} & \cdots & \frac{\partial x_{n;t}}{\partial x_{2;t_0}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_{1;t}}{\partial x_{n;t_0}} & \frac{\partial x_{2;t}}{\partial x_{n;t_0}} & \cdots & \frac{\partial x_{n;t}}{\partial x_{n;t_0}} \end{pmatrix}, \quad H_{t_0}^t(X) = \begin{pmatrix} \frac{x_{1;t_0}}{x_{1;t}} \cdot \frac{\partial x_{1;t}}{\partial x_{1;t_0}} & \frac{x_{1;t_0}}{x_{2;t}} \cdot \frac{\partial x_{2;t}}{\partial x_{1;t_0}} & \cdots & \frac{x_{1;t_0}}{x_{n;t}} \cdot \frac{\partial x_{n;t}}{\partial x_{1;t_0}} \\ \frac{x_{2;t_0}}{x_{1;t}} \cdot \frac{\partial x_{1;t}}{\partial x_{2;t_0}} & \frac{x_{2;t_0}}{x_{2;t}} \cdot \frac{\partial x_{2;t}}{\partial x_{2;t_0}} & \cdots & \frac{x_{2;t_0}}{x_{n;t}} \cdot \frac{\partial x_{n;t}}{\partial x_{2;t_0}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{x_{n;t_0}}{x_{1;t}} \cdot \frac{\partial x_{1;t}}{\partial x_{n;t_0}} & \frac{x_{n;t_0}}{x_{2;t}} \cdot \frac{\partial x_{2;t}}{\partial x_{n;t_0}} & \cdots & \frac{x_{n;t_0}}{x_{n;t}} \cdot \frac{\partial x_{n;t}}{\partial x_{n;t_0}} \end{pmatrix},$$

we can obtain $H_{t_0}^t(X) = \text{diag}(x_{1;t_0}, \dots, x_{n;t_0}) J_{t_0}^t(X) \text{diag}(\frac{1}{x_{1;t}}, \dots, \frac{1}{x_{n;t}})$.

Lemma 3.1. $H_u^v(X)H_v^w(X) = H_u^w(X)$ for any $u, v, w \in \mathbb{T}_n$. In particular, $H_u^v(X)^{-1} = H_v^u(X)$.

Proof. We can view $x_{j;w}$ as a rational function in $x_{1;v}, \dots, x_{n;v}$, and view $x_{k;v}$ as a rational function in $x_{1;u}, \dots, x_{n;u}$, where $j, k = 1, \dots, n$. Thus $\frac{\partial x_{j;w}}{\partial x_{i;u}} = \sum_{k=1}^n \frac{\partial x_{j;w}}{\partial x_{k;v}} \cdot \frac{\partial x_{k;v}}{\partial x_{i;u}} = \sum_{k=1}^n \frac{\partial x_{k;v}}{\partial x_{i;u}} \cdot \frac{\partial x_{j;w}}{\partial x_{k;v}}$, i.e. $J_u^w(X) = J_u^v(X)J_v^w(X)$. It follows that

$$\begin{aligned} & H_u^v(X)H_v^w(X) \\ &= \text{diag}\{x_{1;u}, \dots, x_{n;u}\} J_u^v(X) \text{diag}\{\frac{1}{x_{1;v}}, \dots, \frac{1}{x_{n;v}}\} \cdot \text{diag}\{x_{1;v}, \dots, x_{n;v}\} J_v^w(X) \text{diag}\{\frac{1}{x_{w;1}}, \dots, \frac{1}{x_{w;n}}\} \\ &= \text{diag}\{x_{1;u}, \dots, x_{n;u}\} J_u^v(X) J_v^w(X) \text{diag}\{\frac{1}{x_{1;w}}, \dots, \frac{1}{x_{n;w}}\} \\ &= \text{diag}\{x_{1;u}, \dots, x_{n;u}\} J_u^w(X) \text{diag}\{\frac{1}{x_{1;w}}, \dots, \frac{1}{x_{n;w}}\} \\ &= H_u^w(X). \end{aligned}$$

□

Lemma 3.2. If $\Sigma_v = \mu_k(\Sigma_u)$ for $k \in \langle n \rangle$, then $H_u^v(X)(B_v R^{-1} S^{-1})H_u^v(X)^\top = B_u R^{-1} S^{-1}$, and $\det(H_u^v(X)) = -1$.

Proof. We assume that $\Sigma_u = \Sigma$, $\Sigma_v = \bar{\Sigma}$, $x_{i;u} = x_i$, $x_{i;v} = \bar{x}_i$ for $i \in \langle n \rangle$ and we denote $H = H_u^v(X)$. We have

$$\bar{x}_i = \begin{cases} x_i, & \text{if } i \neq k, \\ x_k^{-1} \left(\prod_{j=1}^n x_j^{[-b_{jk}]_+} \right)^{r_k} \frac{\sum_{m=0}^{r_k} z_{k,m} \hat{y}_k^m}{\bigoplus_{m=0}^{r_k} z_{k,m} y_k^m}, & \text{if } i = k. \end{cases}$$

For $l = k$, $i \neq k$, we have

$$\begin{aligned}
\frac{\partial \bar{x}_k}{\partial x_i} &= \frac{x_k^{-1}}{\bigoplus_{m=0}^{r_k} z_{k,m} y_k^m} \left[\frac{\partial (\prod_{j=1}^n x_j^{[-b_{jk}]_+})^{r_k}}{\partial x_i} \sum_{m=0}^{r_k} z_{k,m} \hat{y}_k^m + (\prod_{j=1}^n x_j^{[-b_{jk}]_+})^{r_k} \frac{\partial (\sum_{m=0}^{r_k} z_{k,m} \hat{y}_k^m)}{\partial x_i} \right] \\
&= r_k [-b_{ik}]_+ x_i^{-1} \bar{x}_k + \frac{x_i^{-1} x_k^{-1}}{\bigoplus_{m=0}^{r_k} z_{k,m} y_k^m} (\prod_{j=1}^n x_j^{[-b_{jk}]_+})^{r_k} \sum_{m=0}^{r_k} m z_{k,m} \hat{y}_k^m b_{ik} \\
\text{thus } \frac{x_i}{\bar{x}_k} \frac{\partial \bar{x}_k}{\partial x_i} &= [-b_{ik}]_+ r_k + \frac{\sum_{m=0}^{r_k} m z_{k,m} \hat{y}_k^m b_{ik}}{\sum_{m=0}^{r_k} z_{k,m} \hat{y}_k^m}.
\end{aligned}$$

We have

$$(4) \quad H_{il} = \frac{x_i}{\bar{x}_l} \frac{\partial \bar{x}_l}{\partial x_i} = \begin{cases} \delta_{il}, & \text{if } l \neq k, \\ -1, & \text{if } i = l = k, \\ [-b_{ik}]_+ r_k + \frac{\sum_{m=0}^{r_k} m z_{k,m} \hat{y}_k^m b_{ik}}{\sum_{m=0}^{r_k} z_{k,m} \hat{y}_k^m}, & \text{if } i \neq k, l = k. \end{cases}$$

It is easy to see that $\det(H) = -1$. Without lose of generality, we may assume $k = 1$. Let

$$\beta = \begin{pmatrix} \bar{b}_{21} \\ \bar{b}_{31} \\ \vdots \\ \bar{b}_{n1} \end{pmatrix}, \quad \alpha = \begin{pmatrix} a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix}, \quad \text{where } a_i = [-b_{i1}]_+ r_1 + \frac{\sum_{m=0}^{r_1} m z_{1,m} \hat{y}_1^m b_{i1}}{\sum_{m=0}^{r_1} z_{1,m} \hat{y}_1^m}. \quad \text{Then } H = \begin{pmatrix} -1 & 0 \\ \alpha & I_{n-1} \end{pmatrix}.$$

$$\text{Let } R = \begin{pmatrix} r_1 & 0 \\ 0 & R_1 \end{pmatrix}, S = \begin{pmatrix} s_1 & 0 \\ 0 & S_1 \end{pmatrix}, \text{ where } R_1 = \text{diag}\{r_2, r_3, \dots, r_n\}, S_1 = \text{diag}\{s_2, s_3, \dots, s_n\}.$$

We can write $B_v = \begin{pmatrix} 0 & \gamma^\top \\ \beta & B_1 \end{pmatrix}$, where $\gamma^\top = -s_1^{-1} r_1^{-1} \beta^\top R_1 S_1$, due to the fact that SRB_v is skew-symmetric. So, for $B = B_u$, $\bar{B} = B_v$, we need only to show that $B_u = HB_v R^{-1} S^{-1} H^\top SR$.

$$\begin{aligned}
&\text{Denote } M = HB_v R^{-1} S^{-1} H^\top SR. \text{ Then, } M = \\
&\begin{pmatrix} -1 & 0 \\ \alpha & I_{n-1} \end{pmatrix} \begin{pmatrix} 0 & -s_1^{-1} r_1^{-1} \beta^\top R_1 S_1 \\ \beta & B_1 \end{pmatrix} \begin{pmatrix} r_1^{-1} & 0 \\ 0 & R_1^{-1} \end{pmatrix} \begin{pmatrix} s_1^{-1} & 0 \\ 0 & S_1^{-1} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ \alpha & I_{n-1} \end{pmatrix}^\top \begin{pmatrix} s_1 & 0 \\ 0 & S_1 \end{pmatrix} \begin{pmatrix} r_1 & 0 \\ 0 & R_1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & s_1^{-1} r_1^{-1} \beta^\top R_1 S_1 \\ -\beta & B_1 + s_1^{-1} r_1^{-1} \beta \alpha^\top S_1 R_1 - s_1^{-1} r_1^{-1} \alpha \beta^\top S_1 R_1 \end{pmatrix}.
\end{aligned}$$

Thus, we can obtain by replacing the entries that

$$\begin{aligned}
M_{il} &= \begin{cases} -\bar{b}_{il}, & \text{if } i = 1 \text{ or } l = 1, \\ \bar{b}_{il} + r_1^{-1} s_1^{-1} \bar{b}_{i1} \left([-b_{l1}]_+ r_1 + \frac{\sum_{m=0}^{r_1} m z_{1,m} \hat{y}_1^m b_{l1}}{\sum_{m=0}^{r_1} z_{1,m} \hat{y}_1^m} \right) s_l r_l - \\ \quad r_1^{-1} s_1^{-1} \left([-b_{i1}]_+ r_1 + \frac{\sum_{m=0}^{r_1} m z_{1,m} \hat{y}_1^m b_{i1}}{\sum_{m=0}^{r_1} z_{1,m} \hat{y}_1^m} \right) \bar{b}_{l1} s_l r_l, & \text{otherwise.} \end{cases} \\
&= \begin{cases} -\bar{b}_{il}, & \text{if } i = 1 \text{ or } l = 1, \\ \bar{b}_{il} + s_1^{-1} s_l r_l \bar{b}_{i1} [-b_{l1}]_+ - s_1^{-1} s_l r_l [-b_{i1}]_+ \bar{b}_{l1}, & \text{otherwise.} \end{cases}
\end{aligned}$$

By (3), we have $\bar{b}_{i1} = -b_{i1}$, $\bar{b}_{l1} = -b_{l1}$. Because SRB and $SR\bar{B}$ are skew-symmetric, we have $s_1 r_1 b_{1l} = -b_{1l} s_l r_l$, $s_1 r_1 \bar{b}_{1l} = -\bar{b}_{1l} s_l r_l$, thus $s_1 r_1 [b_{1l}]_+ = [-b_{1l}]_+ s_l r_l$, and

$$M_{il} = \begin{cases} -\bar{b}_{il}, & \text{if } i = 1 \text{ or } l = 1, \\ \bar{b}_{il} + r_1 (\bar{b}_{i1}[-\bar{b}_{1l}]_+ + [\bar{b}_{i1}]_+ \bar{b}_{1l}), & \text{otherwise.} \end{cases}$$

So, by mutation of \bar{B} , we have $M = \mu_1(\bar{B})$, but $\mu_1(\bar{B}) = B$, so we get $M = B$. \square

$$\text{Let } \epsilon \in \{+, -\}, E_k^\epsilon = (e_{il})_{n \times n}, e_{il} = \begin{cases} \delta_{il}, & \text{if } l \neq k, \\ -1, & \text{if } i = l = k, \\ [\epsilon b_{ik}]_+ r_k, & \text{otherwise.} \end{cases} \text{ We know } (E_k^\epsilon)^2 = I_n.$$

Corollary 3.3. *Keep the above notations. For a seed Σ of M , if $\bar{\Sigma} = \mu_k(\Sigma)$, then $E_k^\epsilon \bar{B} R^{-1} S^{-1} E_k^\epsilon{}^\top = BR^{-1} S^{-1}$.*

Proof. Let $I_k^+ = \{i \mid b_{ik} \geq 0\}$, and $H_0 = H|_{x_i=t, \forall i \in I_k^+}$. We know

$$\begin{aligned} \lim_{t \rightarrow +\infty} \left(([-b_{ik}]_+ r_k + \frac{\sum_{m=0}^{r_k} m z_{k,m} \hat{y}_k^m b_{ik}}{\sum_{m=0}^{r_k} z_{k,m} \hat{y}_k^m}) \Big|_{x_i=t, \forall i \in I_k^+} \right) &= [b_{ik}]_+ r_k \\ \lim_{t \rightarrow +0} \left(([-b_{ik}]_+ r_k + \frac{\sum_{m=0}^{r_k} m z_{k,m} \hat{y}_k^m b_{ik}}{\sum_{m=0}^{r_k} z_{k,m} \hat{y}_k^m}) \Big|_{x_i=t, \forall i \in I_k^+} \right) &= [-b_{ik}]_+ r_k. \end{aligned}$$

Thus $\lim_{t \rightarrow +\infty} H_0 = E_k^+$, $\lim_{t \rightarrow 0} H_0 = E_k^-$. By Lemma 3.2, we have $E_k^\epsilon \bar{B} R^{-1} S^{-1} E_k^\epsilon{}^\top = BR^{-1} S^{-1}$. \square

Remark 3.4. *By corollary 3.3, we obtain $\mu_k(B) = E_k^\epsilon BR^{-1} S^{-1} (E_k^\epsilon)^\top SR$, which was proved for standard cluster pattern in [3].*

Let t_0, t be two vertices in \mathbb{T}_n with a walk

$$t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} t_2 \xrightarrow{k_3} \cdots t_{m-1} \xrightarrow{k_m} t_m = t$$

connecting t_0 and t in \mathbb{T}_n . Write Σ_{t_j} the seed corresponding to t_j , $j = 1, 2, \dots, m$.

Now we can give the useful formula as follows:

Theorem 3.5. (Cluster Formula) *Keep the above notations. It holds that*

$$H_{t_0}^t(X) (B_t R^{-1} S^{-1}) H_{t_0}^t(X)^\top = B_{t_0} R^{-1} S^{-1}, \quad \text{and} \quad \det H_{t_0}^t(X) = (-1)^m.$$

Proof. By Lemma 3.1, $H_{t_0}^t(X) = H_{t_0}^{t_m}(X) = H_{t_0}^{t_1}(X) H_{t_1}^{t_2}(X) \cdots H_{t_{m-1}}^{t_m}(X)$. By Lemma 3.2,

$$\begin{aligned} & H_{t_0}^t(X) (B_t R^{-1} S^{-1}) H_{t_0}^t(X)^\top \\ &= H_{t_0}^{t_1}(X) H_{t_1}^{t_2}(X) \cdots H_{t_{m-1}}^{t_m}(X) (B_{t_m} R^{-1} S^{-1}) H_{t_{m-1}}^{t_m}(X)^\top H_{t_{m-2}}^{t_{m-1}}(X)^\top \cdots H_{t_0}^{t_1}(X)^\top \\ &= H_{t_0}^{t_1}(X) H_{t_1}^{t_2}(X) \cdots H_{t_{m-2}}^{t_{m-1}}(X) (B_{t_{m-1}} R^{-1} S^{-1}) H_{t_{m-2}}^{t_{m-1}}(X)^\top H_{t_{m-3}}^{t_{m-2}}(X)^\top \cdots H_{t_0}^{t_1}(X)^\top \\ &= \cdots = H_{t_0}^{t_1}(X) (B_{t_1} R^{-1} S^{-1}) H_{t_0}^{t_1}(X)^\top = B_{t_0} R^{-1} S^{-1}. \end{aligned}$$

And, $\det H_{t_0}^t(X) = \det H_{t_0}^{t_1}(X) \det H_{t_1}^{t_2}(X) \cdots \det H_{t_{m-1}}^{t_m}(X) = (-1)^m$. \square

Corollary 3.6. *rank(B_t) = rank(B_{t_0}) and $\det(B_t) = \det(B_{t_0})$.*

Corollary 3.7. *If $X_t = X_{t_0}$, then $B_t = B_{t_0}$.*

Proof. If $X_t = X_{t_0}$, then $x_{i;t} = x_{i;t_0}$. In this case, $H_{t_0}^t(X) = I_n$, so we have $B_t = B_{t_0}$. \square

Example 3.8. Consider $S = \text{diag}\{1, 2\}$, $R = \text{diag}\{2, 1\}$, $Z = (z_{1,1})$, assume that M is an (R, Z) -cluster pattern via initial seed $\Sigma_{t_0} = (X_{t_0}, Y_{t_0}, B_{t_0})$ with R -skew-balance S , where $X_{t_0} = (x_1, x_2)$, $Y_{t_0} = (y_1, y_2)$, $B_{t_0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $\Sigma_t = \mu_2 \mu_1(\Sigma_{t_0})$, we have

$$x_{1;t} = \frac{1 + z\hat{y}_1 + \hat{y}_2^2}{x_1(1 \oplus zy_1 \oplus y_1^2)}, \quad x_{2;t} = \frac{1 + \hat{y}_2 + z\hat{y}_1\hat{y}_2 + \hat{y}_1^2\hat{y}_2}{x_2(1 \oplus y_2 \oplus zy_1y_2 \oplus y_1^2y_2)}, \quad B_t = B_{t_0},$$

where $\hat{y}_1 = y_1x_2$, $\hat{y}_2 = y_2x_1^{-1}$. Thus

$$H_{t_0}^t(X) = \begin{pmatrix} -1 & \frac{-y_2 - zy_1y_2x_2 - y_1^2y_2x_2^2}{x_1 + y_2 + zy_1y_2x_2 + y_1^2y_2x_2^2} \\ \frac{zy_1x_2 + 2y_1^2x_2^2}{1 + zy_1x_2 + y_1^2x_2^2} & \frac{y_1^2y_2x_2^2 - x_1 - y_2}{x_1 + y_2 + zy_1y_2x_2 + y_1^2y_2x_2^2} \end{pmatrix}.$$

It is easy to check that $H_{t_0}^t(X)(B_tR^{-1}S^{-1})H_{t_0}^t(X)^T = B_{t_0}R^{-1}S^{-1}$ and $\det(H_{t_0}^t(X)) = 1$.

More information on this example can be seen at Example 2.3 of [17].

3.2. Connection between cluster formula and compatible 2-forms.

In [13, 14], M. Gekhtman, M. Shapiro and A. Vainshtein defined a closed differential 2-form ω compatible with a skew-symmerizable cluster algebra and proved such 2-form always exists for a cluster algebra of geometric type if its exchange matrices have no zero rows. In this part, we give the connection between compatible 2-forms and the cluster formula, and in particular, we prove the compatible 2-form always exists for any (R, Z) cluster pattern.

Definition 3.9. A closed rational differential 2-form ω on an n -affine space is **compatible with the** (R, Z) -cluster pattern M if for any cluster $X = (x_1, \dots, x_n)$ one has $\omega = \sum_{i=1}^n \sum_{j=1}^n w_{ij} \frac{dx_i}{x_i} \wedge \frac{dx_j}{x_j}$, with $w_{ij} \in \mathbb{Q}$. The matrix $\Omega = (w_{ij})$ is called the **coefficient matrix** of ω with respect to X .

Trivially, the coefficient matrix Ω is skew-symmetric.

Theorem 3.10. Let M be an (R, Z) -cluster pattern with any coefficients.

(i) A closed rational differential 2-form ω on an n -affine space is compatible with M if and only if there exists a family of skew-symmetric matrices $\{\Omega_t \in \mathbb{Q}^{n \times n} | t \in \mathbb{T}_n\}$ such that for any $t_0, t \in \mathbb{T}_n$, we have $H_{t_0}^t(X)\Omega_t H_{t_0}^t(X)^\top = \Omega_{t_0}$.

(ii) In particular, there always exists a closed rational differential 2-form ω compatible with M .

Proof. (i): “ \implies ”: Assume that ω is a closed rational differential 2-form on the n -affine space compatible with M , $\Omega_t = (w_{ij}^t)$ is the coefficient matrix of ω with respect to X_t . We know $dx_{i;t} = \sum_{k=1}^n \frac{\partial x_{i;t}}{\partial x_{k;t_0}} dx_{k;t_0}$, thus $\frac{dx_{i;t}}{x_{i;t}} = \sum_{k=1}^n \frac{1}{x_{i;t}} \cdot \frac{\partial x_{i;t}}{\partial x_{k;t_0}} dx_{k;t_0}$, and

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n w_{ij}^t \frac{dx_{i;t}}{x_{i;t}} \wedge \frac{dx_{j;t}}{x_{j;t}} &= \sum_{i=1}^n \sum_{j=1}^n w_{ij}^t \left(\sum_{k=1}^n \frac{1}{x_{i;t}} \cdot \frac{\partial x_{i;t}}{\partial x_{k;t_0}} dx_{k;t_0} \right) \wedge \left(\sum_{l=1}^n \frac{1}{x_{j;t}} \cdot \frac{\partial x_{j;t}}{\partial x_{l;t_0}} dx_{l;t_0} \right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{l=1}^n w_{ij}^t \frac{x_{k;t_0}}{x_{i;t}} \frac{\partial x_{i;t}}{\partial x_{k;t_0}} \cdot \frac{x_{l;t_0}}{x_{j;t}} \frac{\partial x_{j;t}}{\partial x_{l;t_0}} \frac{dx_{k;t_0}}{x_{k;t_0}} \wedge \frac{dx_{l;t_0}}{x_{l;t_0}} \\ &= \sum_{k=1}^n \sum_{l=1}^n \left(\sum_{i=1}^n \sum_{j=1}^n w_{ij}^t \frac{x_{k;t_0}}{x_{i;t}} \frac{\partial x_{i;t}}{\partial x_{k;t_0}} \cdot \frac{x_{l;t_0}}{x_{j;t}} \frac{\partial x_{j;t}}{\partial x_{l;t_0}} \right) \frac{dx_{k;t_0}}{x_{k;t_0}} \wedge \frac{dx_{l;t_0}}{x_{l;t_0}}. \end{aligned}$$

Since

$$\omega = \sum_{i=1}^n \sum_{j=1}^n w_{ij}^t \frac{dx_{i;t}}{x_{i;t}} \wedge \frac{dx_{j;t}}{x_{j;t}} = \sum_{k=1}^n \sum_{l=1}^n w_{kl}^{t_0} \frac{dx_{k;t_0}}{x_{k;t_0}} \wedge \frac{dx_{l;t_0}}{x_{l;t_0}},$$

we have

$$w_{kl}^{t_0} = \sum_{i=1}^n \sum_{j=1}^n w_{ij}^{t_0} \frac{x_{k;t_0}}{x_{i;t_0}} \frac{\partial x_{i;t_0}}{\partial x_{k;t_0}} \cdot \frac{x_{l;t_0}}{x_{j;t_0}} \frac{\partial x_{j;t_0}}{\partial x_{l;t_0}},$$

that is,

$$\Omega_{t_0} = H_{t_0}^t(X) \Omega_t H_{t_0}^t(X)^\top.$$

“ \Leftarrow ”: Assume that $\{\Omega_t \in \mathbb{Q}^{n \times n} | t \in \mathbb{T}_n\}$ is a set of skew-symmetric matrices, satisfying

$$H_{t_0}^t(X) \Omega_t H_{t_0}^t(X)^\top = \Omega_{t_0},$$

for any $t_0, t \in \mathbb{T}_n$, let $\omega = \sum_{i=1}^n \sum_{j=1}^n w_{ij}^{t_0} \frac{dx_{i;t_0}}{x_{i;t_0}} \wedge \frac{dx_{j;t_0}}{x_{j;t_0}}$, be a closed rational differential 2-form on an n -affine space. Replacing $dx_{i;t_0} = \sum_{k=1}^n \frac{\partial x_{i;t_0}}{\partial x_{k;t_0}} dx_{k;t_0}$ into ω , we can see that ω is compatible with M .

(ii): By Theorem 3.5, $\{B_t R^{-1} S^{-1} \in \mathbb{Q}^{n \times n} | t \in \mathbb{T}_n\}$ is a family of skew symmetric matrices satisfying $H_{t_0}^t(X) (B_t R^{-1} S^{-1}) H_{t_0}^t(X)^\top = B_{t_0} R^{-1} S^{-1}$. Let $\Omega_t = B_t R^{-1} S^{-1}$. Then by (i), there always exists a closed rational differential 2-form ω compatible with M . \square

4. ANSWER TO CONJECTURE 1.1 FOR GENERALIZED CLUSTER ALGEBRAS

4.1. On Conjecture 1.1(b) in case of weak geometric type. .

In this section, we firstly prove Conjecture 1.1 (b) for the generalized cluster patterns with coefficients of weak geometric type by using the cluster formula (see Theorem 4.12). The corresponding result for (R, Z) -cluster pattern with coefficients in general semimfield \mathbb{P} , will be studied in the second part of this section, using the theory of D -matrix pattern. Before proving Theorem 4.12, we need some preparations.

Theorem 4.1. (Theorem 2.5 of [5]) *For any (R, Z) -cluster pattern with coefficients in \mathbb{P} , each cluster variable $x_{i;t}$ can be expressed as a Laurent polynomial in $\mathbb{Z}\mathbb{P}[X_{t_0}^{\pm 1}]$.*

Definition 4.2. *Let M be an (R, Z) -cluster pattern with principle coefficients at t_0 , by the Laurent property, each cluster variable $x_{i;t}$ is expressed as a Laurent polynomial $X_{i;t} \in \mathbb{Z}\mathbb{P}[X_{t_0}^{\pm 1}]$, called the **X-function** of $x_{i;t}$, where $\mathbb{P} = \text{Trop}(Y_{t_0}, Z)$.*

Definition 4.3. *The **F-polynomial** of $x_{i;t}$ is defined by $F_{i;t} = X_{i;t}|_{x_{1;t_0}=\dots=x_{n;t_0}=1} \in \mathbb{Z}[Y_{t_0}, Z]$.*

Proposition 4.4. (Proposition 3.3 of [17]) *We have $X_{i;t} \in \mathbb{Z}[X_{t_0}^{\pm 1}, Y_{t_0}, Z]$.*

Proposition 4.5. (Proposition 3.19 and Theorem 3.20 of [17]) *Each **F-polynomial** has constant term 1.*

Corollary 4.6. *Let M be an (R, Z) -cluster pattern with principle coefficients at t_0 , if x is a cluster variable in M , then $-x$ can not be a cluster variable in M .*

Proof. If both x and $-x$ are cluster variables in M , assume that F is the **F-polynomial** corresponding to x , then $-F$ is the **F-polynomial** corresponding to $-x$. This will contradict to that each **F-polynomial** has constant term 1. \square

Let M be an (R, Z) -cluster pattern with principle coefficients and initial seed $\Sigma = (X, Y, B)$, the author in [17] introduced a \mathbb{Z}^n -grading of $\mathbb{Z}[X^{\pm 1}, Y, Z]$ as follows:

$$\deg(x_i) = \mathbf{e}_i, \deg(y_i) = -\mathbf{b}_i, \deg(z_{i;m}) = 0,$$

where \mathbf{e}_i is the i th column vector of I_n , and \mathbf{b}_i is the i th column vector of the initial exchange matrix B . Note that these degrees are vectors in \mathbb{Z}^n .

In [17], the author proved that the X -functions are homogeneous with respect to the \mathbb{Z}^n -grading, and thanks to this, the **g-vector** of a cluster variable $x_{i;t}$ is defined to be the degree of its X -function $X_{i;t}$. From this definition, we have $\deg(X_{i;t}) = (g_{1i}^t, g_{2i}^t, \dots, g_{ni}^t)^\top \in \mathbb{Z}^n$.

Theorem 4.7. (Theorem 3.22 and Theorem 3.23 of [17]) Let M be an (R, Z) -cluster pattern with coefficients in \mathbb{P} and initial seed at t_0 , M_{pr} be the corresponding (R, Z) -cluster pattern with initial principle coefficients at t_0 , which has the same initial cluster and exchange matrix with M , then for the cluster variables $x_{i;t}$ and the coefficients $y_{i;t}$ of M at t , it holds that

$$(5) \quad y_{i;t} = \prod_{j=1}^n y_{j;t_0}^{c_{ji}^t} \prod_{j=1}^n \left(F_{j;t} |_{\mathbb{P}(Y_{t_0}, Z)} \right)^{b_{ji}^t};$$

$$(6) \quad x_{i;t} = \left(\prod_{j=1}^n x_{j;t_0}^{g_{ji}^t} \right) \frac{F_{i;t} |_{\mathcal{F}(\hat{Y}_{t_0}, Z)}}{F_{i;t} |_{\mathbb{P}(Y_{t_0}, Z)}}.$$

Proposition 4.8. Assume that M is an (R, Z) -cluster pattern with principle coefficients at t_0 , and S is the R -skew-balance of M . Let $G_t = (g_{ij}^t)_{n \times n}$. Then we have $G_t = H_{t_0}^t(X) |_{Y_{t_0}=0}$.

Proof. By Theorem 4.7, we know

$$\begin{aligned} x_{j;t} &= F_{j;t}(\hat{Y}_{t_0}, Z) x_{1;t_0}^{g_{1j}^t} \cdots x_{n;t_0}^{g_{nj}^t}, \\ \frac{\partial x_{j;t}}{\partial x_{i;t_0}} &= \frac{g_{ij}^t}{x_{i;t_0}} (x_{1;t_0}^{g_{1j}^t} \cdots x_{n;t_0}^{g_{nj}^t}) F_{j;t}(\hat{Y}_{t_0}, Z) + x_{1;t_0}^{g_{1j}^t} \cdots x_{n;t_0}^{g_{nj}^t} \sum_{k=1}^n \frac{\partial F_{j;t}(\hat{Y}_{t_0}, Z)}{y_k} \frac{b_{ik}^{t_0}}{x_{t_0}} \hat{y}_k. \end{aligned}$$

Thus $\frac{x_{i;t_0}}{x_{j;t}} \frac{\partial x_{j;t}}{\partial x_{i;t_0}} = g_{ij}^t + \sum_{k=1}^n \frac{b_{ik}^{t_0} \hat{y}_k}{F_{j;t}(\hat{Y}_{t_0}, Z)} \frac{\partial F_{j;t}(\hat{Y}_{t_0}, Z)}{y_k},$

By proposition 4.5, it is to make sense to take $Y_{t_0} = 0$ in above equation, so we have

$$g_{ij}^t = \frac{x_{i;t_0}}{x_{j;t}} \frac{\partial x_{j;t}}{\partial x_{i;t_0}} |_{Y_{t_0}=0},$$

i.e. $G_t = H_{t_0}^t(X) |_{Y_{t_0}=0}$. □

From this result and the cluster formula, it is easy to see that $G_t(B_t R^{-1} S^{-1}) G_t^\top = B_{t_0} R^{-1} S^{-1}$ and $\det(G_t) = \pm 1$, as obtained in [17].

Assume that M is an (R, Z) -cluster pattern of weak geometric type with initial seed $(X_{t_0}, Y_{t_0}, B_{t_0})$. Using the notations in Proposition 2.11 and assuming S an R -skew-balance of M , we define \tilde{M} a (\tilde{R}, \tilde{Z}) -cluster pattern with \tilde{R} -skew-balance \tilde{S} and trivial coefficients, given by $(\tilde{X}_{t_0}, \tilde{B}_{t_0})$, where $\tilde{R} = \text{diag}\{R, I_h\}$, $\tilde{Z} = Z$, $\tilde{X}_{t_0} = (\tilde{x}_{1;t_0}, \dots, \tilde{x}_{h+n;t_0})$, with $\tilde{x}_{i;t_0} = x_{i;t_0}$, $\tilde{x}_{n+j;t_0} = y_{j;t_0}$ for $i = 1, \dots, n$, $j = 1, 2, \dots, h$, $\tilde{S} = \text{diag}\{S, I_h\}$, $\tilde{B}_{t_0} = \begin{pmatrix} B_{t_0} & -R^{-1} S^{-1} C_{t_0}^\top \\ C_{t_0} & 0 \end{pmatrix}$.

Clearly, M is a restriction of \tilde{M} from $\{1, \dots, h+n\}$ to $\{1, \dots, n\}$ at t_0 . Assume that $\Sigma_t = \mu_{k_m} \cdots \mu_{k_2} \mu_{k_1}(\Sigma_{t_0})$, where $1 \leq k_j \leq n, j = 1, 2, \dots, m$, then $\tilde{\Sigma}_t = \mu_{k_m} \cdots \mu_{k_2} \mu_{k_1}(\tilde{\Sigma}_{t_0})$. Since $1 \leq k_j \leq n, j = 1, 2, \dots, m$, we can write $H_{t_0}^t(\tilde{X}) = \begin{pmatrix} H_{t_0}^t(X) & 0 \\ H_t & I_h \end{pmatrix}$, $\tilde{B}_t = \begin{pmatrix} B_t & -S^{-1} R^{-1} C_t^\top \\ C_t & \tilde{B}_t \end{pmatrix}$.

Proposition 4.9. Keep the above notations, it holds $SR(H_t B_t + C_t) R^{-1} S^{-1} H_{t_0}^t(X)^\top = C_{t_0}$.

Proof. By Theorem 3.5, we have $H_{t_0}^t(\tilde{X})(\tilde{B}_t\tilde{R}^{-1}\tilde{S}^{-1})H_{t_0}^t(\tilde{X})^\top = \tilde{B}_{t_0}\tilde{R}^{-1}\tilde{S}^{-1}$, thus

$$\begin{aligned} & \begin{pmatrix} H_{t_0}^t(X) & 0 \\ H_t & I_h \end{pmatrix} \begin{pmatrix} B_t & -S^{-1}R^{-1}C_t^\top \\ C_t & \tilde{B}_t \end{pmatrix} \begin{pmatrix} R^{-1} & 0 \\ 0 & I_h \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & I_h \end{pmatrix} \begin{pmatrix} H_{t_0}^t(X) & 0 \\ H_t & I_h \end{pmatrix}^\top \\ &= \begin{pmatrix} B_{t_0} & -R^{-1}S^{-1}C_{t_0}^\top \\ C_{t_0} & 0 \end{pmatrix} \begin{pmatrix} R^{-1} & 0 \\ 0 & I_h \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & I_h \end{pmatrix}. \end{aligned}$$

So we have $SR(H_tB_t + C_t)R^{-1}S^{-1}H_{t_0}^t(X)^\top = C_{t_0}$. \square

Remark 4.10. By Theorem 3.5, $H_{t_0}^t(X)B_tR^{-1}S^{-1}H_{t_0}^t(X)^\top = B_{t_0}R^{-1}S^{-1}$, $\det(H_{t_0}^t(X)) = \pm 1$. Then using this proposition, we can obtain

$$C_t = R^{-1}S^{-1}C_{t_0}(H_{t_0}^t(X)^\top)^{-1}SR - H_t(H_{t_0}^t(X))^{-1}B_{t_0}R^{-1}S^{-1}(H_{t_0}^t(X)^\top)^{-1}SR.$$

Using Proposition 4.9, the following result in [17] can be given directly.

Corollary 4.11. ([17]) Let M be an (R, Z) -cluster pattern with principle coefficients at t_0 and R -skew-balance S , then

$$SRC_tR^{-1}S^{-1}G_t^\top = I_n.$$

Proof. By the definition of H -matrix, and proposition 4.5, we know $H_t|_{Y_{t_0}=0} = 0$. By proposition 4.9 and proposition 4.8, we have $SRC_tR^{-1}S^{-1}G_t^\top = I_n$. \square

Now, we can give the positive affirmation on Conjecture 1.1 (b) in case of weak geometric type.

Theorem 4.12. Assume that M is an (R, Z) -cluster pattern of weak geometric type at t_0 , with an R -skew-balance S . Then for each t , the seed Σ_t is uniquely determined by X_t .

Proof. We know $\Sigma_t = (X_t, Y_t, B_t)$ and Y_t is uniquely determined by C_t . However, by Theorem 3.5, we have

$$(7) \quad B_t = (H_{t_0}^t(X))^{-1}B_{t_0}R^{-1}S^{-1}(H_{t_0}^t(X)^\top)^{-1}SR.$$

By remark 4.10, we have

$$(8) \quad C_t = R^{-1}S^{-1}C_{t_0}(H_{t_0}^t(X)^\top)^{-1}SR - H_t(H_{t_0}^t(X))^{-1}B_{t_0}R^{-1}S^{-1}(H_{t_0}^t(X)^\top)^{-1}SR.$$

We know the right side of (7) and (8) is uniquely determined by X_t , thus B_t and C_t is uniquely determined by X_t , which implies that Σ_t is uniquely determined by X_t . \square

4.2. D -matrix pattern and answer to Conjecture 1.1. .

Let M be an (R, Z) -cluster pattern with coefficients in \mathbb{P} and initial seed $\Sigma_{t_0} = (X_{t_0}, P_{t_0}, B_{t_0})$. By Laurent phenomenon, we can express the cluster variable $x_{i;t}$ in Σ_t , as

$$(9) \quad x_{i;t} = \frac{f_{i;t}(x_{1;t_0}, \dots, x_{n;t_0})}{x_{1;t_0}^{d_{1i}^t} \cdots x_{n;t_0}^{d_{ni}^t}},$$

where $f_{i;t}$ is a polynomial in $x_{1;t_0}, \dots, x_{n;t_0}$ with coefficients in $\mathbb{Z}\mathbb{P}$, such that $x_{j;t_0} \nmid f_{i;t}$.

Define $\mathbf{d}_i^t = (d_{1i}^t, d_{2i}^t, \dots, d_{ni}^t)^\top$ which is called the **d-vector** of $x_{i;t}$.

Define $D_t = (d_{ij}^t) = (\mathbf{d}_1^t, \mathbf{d}_2^t, \dots, \mathbf{d}_n^t)$, called the **D -matrix** of the cluster X_t . Clearly, $D_{t_0} = -I_n$.

Proposition 4.13. D_t is uniquely determined by the initial condition $D_{t_0} = -I_n$, together with the relation as follows under mutation of seeds:

$$(10) \quad (D_{t'})_{ij} = \begin{cases} d_{ij}^t & \text{if } j \neq k; \\ -d_{ik}^t + \max\left\{\sum_{b_{ik}^t > 0} d_{il}^t b_{lk}^t r_k, \sum_{b_{ik}^t < 0} -d_{il}^t b_{lk}^t r_k\right\} & \text{if } j = k. \end{cases}$$

for any $t, t' \in \mathbb{T}_n$ with edge $t \xrightarrow{k} t'$.

Proof. We know $x_{k;t'} = x_{k;t}^{-1} \left(\prod_{j=1}^n x_{j;t}^{[-b_{jk}^t]_+} \right)^{r_k} \frac{\sum_{m=0}^{r_k} z_{k,m} \hat{y}_{k;t}^m}{\bigoplus_{m=0}^{r_k} z_{k,m} y_{k;t}^m}$, where $\hat{y}_{k;t}^m = y_{k;t} \prod_{j=1}^n x_{j;t}^{b_{jk}^t}$. We can obtain $d_{ik}^{t'} = -d_{ik}^t + \max\{ \sum_{b_{ik}^t > 0} d_{il}^t b_{lk}^t r_k, \sum_{b_{ik}^t < 0} -d_{il}^t b_{lk}^t r_k \}$, by (9). \square

In this proposition, the case for standard cluster pattern has been given in [11].

Corollary 4.14. *The \mathbf{d} -vectors of the (R, Z) -cluster pattern with initial seed $(X_{t_0}, Y_{t_0}, B_{t_0})$ coincide with the \mathbf{d} -vectors of the standard cluster pattern with initial seed $(X_{t_0}, Y_{t_0}, B_{t_0} R)$.*

Definition 4.15. A D -matrix pattern W at t_0 is an assignment for each pair $\Delta_t := (D_t, Q_t)$, called a **matrix seed**, to a vertex t of the n -regular tree \mathbb{T}_n with $\Delta_{t_0} = (-I_n, Q_{t_0})$, which is called the **initial matrix seed**, where Q_{t_0} is a skew-symmetrizable matrix. And for any edge $t \xrightarrow{k} t'$, $\Delta_{t'} = (D_{t'}, Q_{t'})$ and $\Delta_t = (D_t, Q_t)$ are related with $Q_{t'} = \mu_k(Q_t)$ by the standard matrix mutation μ_k and $D_{t'}$ is defined satisfying (10) in Proposition 4.13. Denote $\mu_k^{ms}(\Delta_t) := \Delta_{t'}$, where μ_k^{ms} is called the **mutation of matrix seed in the direction k** .

Remark 4.16. By Remark 2.5, any (R, Z) -cluster pattern M with initial seed at t_0 can supply the corresponding D -matrix pattern W with matrix seed $\Delta_t = (D_t, B_t R)$ at $t \in \mathbb{T}_n$, where D_t is the D -matrix of the cluster X_t . This W is called the **D -matrix pattern induced by M at t_0** .

Definition 4.17. (i) For an (R, Z) -cluster pattern M , two seeds $\Sigma_t = (X_t, Y_t, B_t)$ and $\Sigma_{t'} = (X_{t'}, Y_{t'}, B_{t'})$, or say, their corresponding vertices t and t' in \mathbb{T}_n , are called **\mathcal{M} -equivalent** if there exists a permutation $\sigma \in S_n$ such that $x_{i;t'} = x_{\sigma(i);t}$, $y_{i;t'} = y_{\sigma(i);t}$ and $b_{ij}^{t'} = b_{\sigma(i)\sigma(j)}^t$, denote as $\Sigma_t \simeq_{\mathcal{M}} \Sigma_{t'}$.

(ii) For a D -matrix pattern W , two matrix seeds $\Delta_t = (D_t, Q_t)$ and $\Delta_{t'} = (D_{t'}, Q_{t'})$, or say, their corresponding vertices t and t' in \mathbb{T}_n are **\mathcal{W} -equivalent** if there exists a permutation $\sigma \in S_n$ such that $\mathbf{d}_i^{t'} = \mathbf{d}_{\sigma(i)}^t$ and $q_{ij}^{t'} = q_{\sigma(i)\sigma(j)}^t$, denote as $\Delta_t \simeq_{\mathcal{W}} \Delta_{t'}$.

Definition 4.18. The **exchange graph** Γ of a matrix pattern W (respectively, (R, Z) -cluster pattern M) is defined as the graph whose vertices are the \mathcal{W} -equivalence classes of matrix seeds $[\Delta_t]$ (respectively, \mathcal{M} -equivalence classes of seeds $[\Sigma_t]$) and whose edges given between $[\Delta_{t_1}]$ and $[\Delta_{t_2}]$ (respectively, $[\Sigma_{t_1}]$ and $[\Sigma_{t_2}]$) for $t_1, t_2 \in \mathbb{T}_n$ if there exists $k \in \{1, \dots, n\}$ such that $\mu_k^{ms}(\Delta_{t_1}) \in [\Delta_{t_2}]$ (respectively, $\mu_k(\Sigma_{t_1}) \in [\Sigma_{t_2}]$).

Remark 4.19. By the definition, the exchange graph of a D -matrix pattern only depends on the initial exchange matrix Δ_{t_0} .

Now we discuss the further relationship between an (R, Z) -cluster pattern with initial seed Σ_{t_0} and the matrix pattern induced by it at t_0 .

Lemma 4.20. Let M_{pr} be an (R, Z) -cluster pattern with principal coefficients at t_0 . If there exists a permutation $\sigma \in S_n$ such that $\mathbf{d}_{\sigma(i)}^t = \mathbf{d}_i^{t_0}$, where \mathbf{d}_i^t are the i -th columns of D_t for all i , then $x_{\sigma(i);t} = x_{i;t_0}$ and $\Sigma_t \simeq_{\mathcal{M}} \Sigma_{t_0}$.

Proof. By Proposition 4.4, there exist polynomials f_1, \dots, f_n in $x_{1;t_0}, \dots, x_{n;t_0}$ with coefficients in $\mathbb{Z}[Y_{t_0}, Z]$, and $x_{j;t_0} \nmid f_i$ for any i, j in order to get D_t . But $\mathbf{d}_{\sigma(i)}^t = \mathbf{d}_i^{t_0}$ and $D_{t_0} = -I_n$, then

$$(11) \quad x_{\sigma(1);t} = x_{1;t_0} f_1, \dots, x_{\sigma(n);t} = x_{n;t_0} f_n.$$

So we have $x_{i;t_0} = \frac{x_{\sigma(i);t}}{f_i(x_{1;t_0}, \dots, x_{n;t_0})}$. Conversely, there exist polynomials g_1, \dots, g_n in $x_{1;t}, \dots, x_{n;t}$ with coefficients in $\mathbb{Z}[Y_{t_0}, Z]$, and $x_{j;t} \nmid g_i$ for any i, j , such that

$$(12) \quad x_{1;t_0} = \frac{g_1(x_{1;t}, \dots, x_{n;t})}{x_{1;t}^{k_{11}} \dots x_{n;t}^{k_{n1}}}, \dots, x_{n;t_0} = \frac{g_n(x_{1;t}, \dots, x_{n;t})}{x_{1;t}^{k_{1n}} \dots x_{n;t}^{k_{nn}}}.$$

Hence, for any i ,

$$\frac{g_i(x_{1;t}, \dots, x_{n;t})}{x_{1;t}^{k_{1i}} \dots x_{n;t}^{k_{ni}}} = x_{i;t_0} = \frac{x_{\sigma(i);t}}{f_i\left(\frac{g_1(x_{1;t}, \dots, x_{n;t})}{x_{1;t}^{k_{11}} \dots x_{n;t}^{k_{n1}}}, \dots, \frac{g_n(x_{1;t}, \dots, x_{n;t})}{x_{1;t}^{k_{1n}} \dots x_{n;t}^{k_{nn}}}\right)},$$

For the right side of this equality, we can write $x_{i;t_0}$ as that $x_{i;t_0} = x_{1;t}^{\lambda_{1;i}} \dots x_{n;t}^{\lambda_{n;i}} / h_i(x_{1;t}, \dots, x_{n;t})$, where h_i is a polynomial in $x_{1;t}, \dots, x_{n;t}$ such that $x_{j;t} \nmid h_i$ for $j = 1, \dots, n$. So we have

$$g_i(x_{1;t}, \dots, x_{n;t}) h_i(x_{1;t}, \dots, x_{n;t}) = x_{1;t}^{k_{1;i} + \lambda_{1;i}} \dots x_{n;t}^{k_{n;i} + \lambda_{n;i}},$$

However, due to $x_{j;t} \nmid g_i$ and $x_{j;t} \nmid h_i$ for $j = 1, \dots, n$, it implies that $g_i = \pm 1 = h_i$, then from (12), we have $x_{i;t_0} = \frac{\pm 1}{x_{1;t}^{k_{1i}} \dots x_{n;t}^{k_{ni}}}$. From this and by the definition of $H_t^{t_0}(X)$, we can obtain that $H_t^{t_0}(X) = (-k_{ij})_{n \times n}$. By Theorem 3.5, $\det H_t^{t_0}(X) = \pm 1$. By Lemma 3.1, $H_t^{t_0}(X) = H_t^{t_0}(X)^{-1}$. Then we have $H_t^{t_0}(X) \in M_n(\mathbb{Z})$.

Lemma 4.21. *For any $i = 1, \dots, n$, $x_{i;t}$ is a Laurent monomial in $x_{1;t_0}, \dots, x_{n;t_0}$ and $f_i = \pm 1$.*

Proof. Without loss of generality, we can assume that $i = 1$. By (11) and the definition of $H_t^{t_0}(X)_{j\sigma(1)}$, we can get $H_t^{t_0}(X)_{j\sigma(1)} = \delta_{1j} + \frac{x_{j;t_0}}{f_1} \frac{\partial f_1}{\partial x_{j;t_0}}$. Since $H_t^{t_0}(X) \in M_n(\mathbb{Z})$, $\frac{x_{j;t_0}}{f_1} \frac{\partial f_1}{\partial x_{j;t_0}}$ is an integer. Write $f_1 = a_m x_{j;t_0}^m + a_{m-1} x_{j;t_0}^{m-1} + \dots + a_1 x_{j;t_0} + a_0$, where $a_m \neq 0$ and a_0, \dots, a_m are polynomials of $x_{1;t_0}, \dots, x_{j-1;t_0}, x_{j+1;t_0}, \dots, x_{n;t_0}$ with coefficients in $\mathbb{Z}[Y_{t_0}, Z]$, then $\frac{x_{j;t_0}}{f_1} \frac{\partial f_1}{\partial x_{j;t_0}} = \frac{m a_m x_{j;t_0}^{m-1} + (m-1) a_{m-1} x_{j;t_0}^{m-2} + \dots + a_1}{a_m x_{j;t_0}^m + a_{m-1} x_{j;t_0}^{m-1} + \dots + a_1 x_{j;t_0} + a_0}$ is an integer. If $m > 0$, then $\frac{x_{j;t_0}}{f_1} \frac{\partial f_1}{\partial x_{j;t_0}} = m$ and $a_0 = a_1 = \dots = a_{m-1} = 0$. So $f_1 = a_m x_{j;t_0}^m$, which contradicts to $x_{j;t} \nmid f_1$. Thus $m = 0$ and $f_1 = a_0$, which is a polynomial of $x_{1;t_0}, \dots, x_{j-1;t_0}, x_{j+1;t_0}, \dots, x_{n;t_0}$ with coefficients in $\mathbb{Z}[Y_{t_0}, Z]$. Since j can take value from 1 to n , f_1 must be in $\mathbb{Z}[Y_{t_0}, Z]$. By (11), we have $x_{1;t_0} = \frac{x_{\sigma(1);t}}{f_1}$, then by Proposition 4.4, $\frac{1}{f_1} \in \mathbb{Z}[Y_{t_0}, Z]$, which means $f_1 = \pm 1$. \square

Return to the proof of Lemma 4.20. By Lemma 4.21 and (11), we have $x_{\sigma(i);t} = \pm x_{i;t_0}$. Thus $x_{\sigma(i);t} = x_{i;t_0}$ by Corollary 4.6. It is easy to see that $b_{\sigma(i)\sigma(j)}^t = b_{ij}^{t_0}$, $\mathbf{g}_{\sigma(i)}^t = \mathbf{g}_i^{t_0}$ and $\mathbf{c}_{\sigma(i)}^t = \mathbf{c}_i^{t_0}$. Thus $\Sigma_t \simeq_{\mathcal{M}} \Sigma_{t_0}$. \square

Theorem 4.22. *Let M be an (R, Z) -cluster pattern with initial seed $(X_{t_0}, Y_{t_0}, B_{t_0})$ at t_0 , W be the D -matrix pattern induced by M at t_0 . Let M_{pr} be the corresponding (R, Z) -cluster pattern with principal coefficients at t_0 of M , denote by Σ_t^{pr} the seed of M_{pr} at t . Then,*

- (i) *The following statements are equivalent: (1) $\Sigma_{t_1} \simeq_{\mathcal{M}} \Sigma_{t_2}$, (2) $\Sigma_{t_1}^{pr} \simeq_{\mathcal{M}} \Sigma_{t_2}^{pr}$, (3) $\Delta_{t_1} \simeq_{\mathcal{W}} \Delta_{t_2}$;*
- (ii) $\Gamma_M = \Gamma_{M_{pr}} = \Gamma_W$.

Proof. (i): Firstly, we prove (2) \iff (3). “ \implies ”: Obviously.

“ \impliedby ”: Since $\Delta_{t_1} \simeq_{\mathcal{W}} \Delta_{t_2}$, there exists a permutation $\sigma \in S_n$ such that $\mathbf{d}_{\sigma(i)}^{t_1} = \mathbf{d}_{\sigma(i)}^{t_2}$ and $b_{ij}^{t_1} = b_{\sigma(i)\sigma(j)}^{t_2}$. We always have $\Sigma_{t_0}^{pr} = \mu_{i_k} \dots \mu_{i_2} \mu_{i_1}(\Sigma_{t_1}^{pr})$ for a series of mutations $\mu_{i_1}, \mu_{i_2}, \dots, \mu_{i_k}$. Let $\Sigma_t^{pr} = \mu_{\sigma(i_k)} \dots \mu_{\sigma(i_2)} \mu_{\sigma(i_1)}(\Sigma_{t_2}^{pr})$, then we have $\mathbf{d}_i^{t_0} = \mathbf{d}_{\sigma(i)}^{t_1}$ and $b_{ij}^{t_0} = b_{\sigma(i)\sigma(j)}^{t_1}$. By Lemma 4.20, we have $x_{\sigma(i);t}^{pr} = x_{i;t_0}^{pr}$ and $\mathbf{c}_{\sigma(i)}^t = \mathbf{c}_i^{t_0}$. Since $\Sigma_{t_1}^{pr} = \mu_{i_1} \mu_{i_2} \dots \mu_{i_k}(\Sigma_{t_0}^{pr})$ and $\Sigma_{t_2}^{pr} = \mu_{\sigma(i_1)} \mu_{\sigma(i_2)} \dots \mu_{\sigma(i_k)} \mu_{\sigma(i_1)}(\Sigma_{t_0}^{pr})$, we obtain $x_{\sigma(i);t_2}^{pr} = x_{i;t_1}^{pr}$, then $b_{\sigma(i)\sigma(j)}^{t_2} = b_{ij}^{t_1}$ and $\mathbf{c}_{\sigma(i)}^{t_2} = \mathbf{c}_i^{t_1}$, which means $\Sigma_{t_1}^{pr} \simeq_{\mathcal{M}} \Sigma_{t_2}^{pr}$.

(1) \implies (3) is also trivial. Now we prove (2) \implies (1).

If $\Sigma_{t_1}^{pr} \simeq_{\mathcal{M}} \Sigma_{t_2}^{pr}$, then there exists a permutation $\sigma \in S_n$ such that $b_{\sigma(i)\sigma(j)}^{t_1} = b_{ij}^{t_2}$, $\mathbf{c}_{\sigma(i)}^{t_1} = \mathbf{c}_i^{t_2}$, $\mathbf{g}_{\sigma(i)}^{t_1} = \mathbf{g}_i^{t_2}$ and $F_{\sigma(i);t_1} = F_{i;t_2}$. By Theorem 4.7, for cluster variables and coefficients of M , we have the relations $x_{\sigma(i);t_1} = x_{i;t_2}$, $y_{\sigma(i);t_1} = x_{i;t_2}$, then $X_{t_1} = X_{t_2}$, $Y_{t_1} = Y_{t_2}$ as sets. It follows that Σ_{t_1} and Σ_{t_2} are \mathcal{M} -equivalent.

(ii) is obtained directly from (i). \square

From this theorem, we now can answer Conjecture 1.1 (a),(b),(c) for a generalized cluster algebra in the statements (a),(b),(c) respectively as follows.

Theorem 4.23. *Given an (R, Z) -cluster pattern M with coefficients in \mathbb{P} and initial seed Σ_{t_0} , the following statements hold:*

- (a) *The exchange graph only depends on the initial exchange matrix B_{t_0} .*
- (b) *Every seed Σ_t in M is uniquely determined by X_t .*
- (c) *Two clusters are adjacent in the exchange graph Γ_M if and only if they have exactly $n - 1$ common cluster variables.*

Proof. (a) Let W be the D -matrix pattern induced by M at t_0 . By Theorem 4.22, $\Gamma_M \simeq \Gamma_W$. Moreover, the result follows from Remark 4.19.

(b) By Theorem 4.22, Σ_t is uniquely determined by $\Delta_t = (D_t, B_t R)$. And, by the definition of D_t , D_t is uniquely determined by X_t and the initial seed Σ_{t_0} . By Theorem 3.5, B_t is also determined by X_t . Then the result follows.

(c) “ \implies ”: It is clear from the definition of mutation.

“ \impliedby ”: Assume that X_{t_1} and X_{t_2} are two clusters of M with $n - 1$ common cluster variables, we will prove that the matrix seed Δ_{t_1} and Δ_{t_2} are adjacent in the exchange graph Γ_W . Thus, by (b) and Theorem 4.22, X_{t_1} and X_{t_2} are adjacent in the exchange graph Γ_M .

Let M_{tr} be the corresponding (R, Z) -cluster pattern with trivial coefficients of M , which has the same initial exchange matrix with M . The cluster variables $x_{i,t}$ of M_{tr} can be obtained from the corresponding cluster variables $x_{i,t}^o$ of M via valuing their coefficients to 1. Hence, any pair of equal cluster variables in X_{t_1} and X_{t_2} respectively becomes a pair of equal cluster variables in $X_{t_1}^{tr}$ and $X_{t_2}^{tr}$ respectively. Without loss of generality, let $X_{t_1}^{tr} = (x_1, x_2, \dots, x_n)$, $X_{t_2}^{tr} = (w_1, x_2, \dots, x_n)$, $X_{t_3}^{tr} = \mu_{x_1}(X_{t_1}^{tr}) = (\bar{x}_1, x_2, \dots, x_n)$.

If $x_1 = w_1$, then by (b) and Theorem 4.22, $\Delta_{t_1} \simeq_W \Delta_{t_1}$ and then $X_{t_1} = X_{t_2}$ as sets. This is a contradiction. Hence, we have $x_1 \neq w_1$. Then, the given clusters $X_{t_1}^{tr}$ and $X_{t_2}^{tr}$ of M_{tr} have also $n - 1$ common cluster variables.

By the definition of $H_{t_1}^{t_2}(X^{tr})$ and $H_{t_3}^{t_2}(X^{tr})$, they can be written as the form $H_{t_1}^{t_2}(X^{tr}) = \begin{pmatrix} a & O_{1 \times (n-1)} \\ \alpha_{(n-1) \times 1} & I_{n-1} \end{pmatrix}$ and $H_{t_3}^{t_2}(X^{tr}) = \begin{pmatrix} \bar{a} & O_{1 \times (n-1)} \\ \bar{\alpha}_{(n-1) \times 1} & I_{n-1} \end{pmatrix}$. By the cluster formula, $\det H_{t_3}^{t_2}(X^{tr}) = -\det H_{t_1}^{t_2}(X^{tr}) = \pm 1$, thus $\bar{a} = -a = \pm 1$.

If $a = 1$, i.e. $\frac{x_1}{w_1} \frac{\partial w_1}{\partial x_1} = 1$, we will show that $w_1 = x_1$, which is a contradiction.

By Laurent phenomenon, w_1 can be written as

$$(13) \quad w_1 = f(x_1, \dots, x_n) / (x_1^{d_1} \cdots x_n^{d_n}),$$

where f is a polynomial in x_1, \dots, x_n with coefficients in $\mathbb{Z}[Z]$ with $x_i \nmid f$ for any i . By $1 = \frac{x_1}{w_1} \frac{\partial w_1}{\partial x_1} = -d_1 + \frac{x_1}{f} \frac{\partial f}{\partial x_1}$, we know $\frac{x_1}{f} \frac{\partial f}{\partial x_1}$ is an integer. Just as in the proof of Lemma 4.21, we can show that x_1 does not appear in f , i.e. f is a polynomial in x_2, \dots, x_n . Thus $\frac{x_1}{f} \frac{\partial f}{\partial x_1} = 0$ and $d_1 = -1$. Then by (13), we have $x_1 = \frac{w_1}{x_2^{-d_1} \cdots x_n^{-d_n} f(x_2, \dots, x_n)}$. By Laurent phenomenon, it is easy to know f is a monomial in x_2, \dots, x_n . Since $x_i \nmid f$, we obtain $f = 1$ and $w_1 = x_1 x_2^{-d_2} \cdots x_n^{-d_n}$.

For $i \neq 1$, if $d_i > 0$, we consider the cluster variable \bar{x}_i obtained from $X_{t_1}^{tr}$ by mutation at cluster variable x_i . Then $x_i \bar{x}_i = g(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. Clearly, g is a nontrivial polynomial, otherwise, the generalized cluster algebra generated by x_i will split off, which contradicts to that $d_i \neq 0$. Therefore $w_1 = (x_1 x_2^{-d_2} \cdots x_{i-1}^{-d_{i-1}} \bar{x}_i^{d_i} x_{i+1}^{-d_{i+1}} \cdots x_n^{-d_n}) / g^{d_i}$. It contradicts to Laurent phenomenon since g is an exchange polynomial. So $d_i \leq 0$ for $i = 2, \dots, n$.

Consider $x_1 = w_1 x_2^{d_2} \cdots x_n^{d_n}$ and use the similar discussion as above, we can show $d_i \geq 0$ dually for $i = 2, \dots, n$. Thus $d_2 = \cdots = d_n = 0$, and we obtain $w_1 = x_1$. It is impossible. Therefore we have only $a = -1$, then $\bar{a} = 1$.

Since $\bar{a} = 1$, we can repeat the above discussion via replacing $X_{t_1}^{tr}$ by $X_{t_3}^{tr}$, and obtain $w_1 = \bar{x}_1$, i.e. $\mu_{x_1}(X_{t_1}^{tr}) = X_{t_2}^{tr}$. By (b) and the definition of matrix seed, it follows that the matrix seed Δ_{t_1} and Δ_{t_2} are adjacent in the exchange graph Γ_W . Then the result holds. \square

We know that a pattern (cluster pattern or matrix pattern) is said to be of **finite type**, if the exchange graph has finite many vertexes.

Corollary 4.24. *Assume M is an (R, Z) -cluster pattern with initial seed $(X_{t_0}, Y_{t_0}, B_{t_0})$, let \bar{M} be the standard cluster pattern with initial seed $(X_{t_0}, Y_{t_0}, B_{t_0}R)$ at t_0 induced from M , then*

- (i) $\Sigma_{t_1} \simeq_{\mathcal{M}} \Sigma_{t_2}$ if and only if $\bar{\Sigma}_{t_1} \simeq_{\mathcal{M}} \bar{\Sigma}_{t_2}$,
- (ii) $\Gamma_M \simeq \Gamma_{\bar{M}}$.

Proof. We know that M and \bar{M} induce the same D -matrix pattern W at t_0 . By Theorem 4.22, we have $\Sigma_{t_1} \simeq_{\mathcal{M}} \Sigma_{t_2}$ if and only if $\Delta_{t_1} \simeq_{\mathcal{M}} \Delta_{t_2}$ if and only if $\bar{\Sigma}_{t_1} \simeq_{\mathcal{M}} \bar{\Sigma}_{t_2}$, then $\Gamma_M \simeq \Gamma_W \simeq \Gamma_{\bar{M}}$. \square

Following Corollary 4.24 (ii), we have furthermore:

Corollary 4.25. *M is of finite type if and only if \bar{M} is of finite type.*

Remark 4.26. *The classifications of standard cluster algebras and generalized cluster algebras of finite type has been given respectively in [9] and [5]. Corollary 4.25 actually supplies a simple way to give the classification of generalized cluster algebras of finite type via that of standard cluster algebras.*

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